Indian Statistical Institute, Bangalore Centre

B.Math (Hons.) II Year, First Semester

Semestral Examination

Algebra III

December 3, 2012 Time: 9.00 AM - 1.00 PM Instructor: Bhaskar Bagchi

Remark : This paper carries a total of 130 marks. The maximum you can score is 100.

1. Let V be an abelian group. Show that V admits at most one structure of $\mathbb{Q}$-module. That is, there is at most one choice of the scalar multiplication $\mathbb{Q} \times V \to V$ which (together with the given addition on V) satisfies the axioms for a Q-module. [20]

2. (a) If R is a commutative ring which is non-trivial, then show that the R-modules R^n, $n = 1, 2, 3, \ldots$ are mutually non-isomorphic.

   (b) Let V be a real vector space with a countably infinite basis $\{v_n : n = 1, 2, \ldots\}$. Let R be the non-commutative ring consisting of all linear transformations from V into V (with usual operations). Let $S, T \in R$ be defined by

$$S(v_n) = \begin{cases} v_{n/2} \text{ if } n \text{ even} \\ 0 \text{ if } n \text{ odd} \end{cases}$$

$$T(v_n) = \begin{cases} v_{\frac{n+1}{2}} \text{ if } n \text{ odd} \\ 0 \text{ if } n \text{ even.} \end{cases}$$

   Show that $\{S, T\}$ is a basis for R as an R-module. Hence conclude that the R-modules $R^n, n = 1, 2, 3, \ldots$ are all isomorphic.

   [10+20=30]

3. (a) Prove that a module M is Noetherian iff all its submodules are finitely generated.

   (b) Show that the set of all algebraic integers in $\mathbb{C}$ constitute a subring of $\mathbb{C}$.

   (c) Show that the ring in (b) is not Noetherian.

   [10+10+10=30]

4. Let R be a commutative ring and I be an ideal in $R[X]$. Put $A = \{x \in R : x = 0$ or x is the leading coefficient of some non-zero element of $I\}$. Show that A is an ideal in R. [20]

5. Let $\omega \neq 1$ be a complex cube root of unity. Let $R = Z[\omega]$.

   (a) Find all the units in R.

(b) Show that R is a euclidean domain.

(c) Find all the prime elements of R.

(Hint : For the last part, you may assume the following fact without proof. If p is an ordinary prime there are integers x, y such that $x \not\equiv 0 \pmod{p}, y \not\equiv 0 \pmod{p}$ but $x^2 + xy + y^2 \equiv 0 \pmod{p}$ iff either $p = 3$ or $p \equiv 1 \pmod{3}$) [10+10+10=30]